EXAM GROUP THEORY,

October 29th, 2024, 3:00pm–5:00pm,

Exam Hall 2 I1 - M20.

Put your name on every sheet of paper you hand in. Please provide complete arguments for each of your answers. The exam consists of 3 open questions and 3 multiple choice problems. You can score up to 18 points for the open questions and here you obtain 2 points for free. For each of the multiple choice problems you can score 3 points, and here 1 point is for free. In this way you will score in total between 2 and 20 points for the open questions. You score between 1 and 10 points for the multiple choice part. Recall: the open questions count 70%, the multiple choice ones 10% and the homework 20% towards the final grade for the course.

(1) Consider the map $\varphi\colon S_5 \to S_5$ that squares any permutation: $\varphi(\tau) = \tau^2$.
   (a) [2 points.] Show that $\varphi(S_5) \subseteq A_5$.
   (b) [2 points.] Show that the converse holds as well: $A_5 \subseteq \varphi(S_5)$.
   (c) [2 points.] Give (with proof!) an element of A_6 that is not the square of any permutation in S_6.

(2) (a) [2 points.] Find (explain your answer!) the number of pairwise non-isomorphic abelian groups of order $2^{10} = 1024$.
   (b) [2 points.] How many of the groups in (a) have precisely three elements of order 2?
   (c) [2 points.] Give (with proof!) an example of a non-commutative group that has order 1024.

(3) Let G be any group. By $\mathrm{Aut}(G)$ we denote the subgroup of the group S_G (the bijections from G to G), consisting of all bijections that are moreover group homomorphisms. We consider the map $\psi\colon G \to \mathrm{Aut}(G)$ defined by $\psi(g) = \gamma_g$ (so ψ maps g to the conjugation by g, i.e., $\psi(g)(x) = gxg^{-1}$).
   (a) [1 point.] Show that ψ is a homomorphism.
   (b) [2 points.] Prove that the kernel of ψ is equal to the center $\mathcal{Z}(G)$ of G.
   (c) [1 point.] Why is $\psi(G)$ a subgroup of $\mathrm{Aut}(G)$?
   (d) [2 points.] Show that the subgroup $\psi(G)$ of $\mathrm{Aut}(G)$ is normal.

PLEASE TURN OVER

(4) AND FINALLY THREE MULTIPLE CHOICE QUESTIONS:

(i) [3 points.] Select the correct one from the following assertions:
  a. $(\mathbb{Z}/42\mathbb{Z})^\times$ contains an element of order 12.
  b. $(\mathbb{Z}/42\mathbb{Z})^\times$ contains an element of order 21.
  c. $(\mathbb{Z}/42\mathbb{Z})^\times$ contains an element of order 4.
  d. $(\mathbb{Z}/42\mathbb{Z})^\times$ contains three elements of order 2.

(ii) [3 points.] Select the incorrect one from the following assertions:
  a. If an integer $n > 0$ is square free (i.e., not divisible by any m^2 with $m \in \mathbb{Z}_{\geq 2}$), then up to isomorphism only one abelian group of order n exists.
  b. If p is a prime number and G is a group of order $2p$, then G has at least 3 normal subgroups.
  c. Any dihedral group of even order $2n$ contains exactly n elements of order 2.
  d. If G is a simple group (i.e., its only normal subgroups are $\{e\}$ and G), then a homomorphism from G to another group is either constant or injective.

(iii) [3 points.] Given is that $\mathbb{Z}^d/H$ is a finite group of order at least 2. Which of the following properties holds?
  a. Some $n \in \mathbb{Z}_{\geq 2}$ exists such that $n \cdot \mathbb{Z}^d \subseteq H$.
  b. No isomorphism $\mathbb{Z}^d \cong H$ can exist.
  c. The factor group $\mathbb{Z}^d/H$ is cyclic.
  d. The order of the factor group $\mathbb{Z}^d/H$ is bounded by $2d$.